Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 14.02.2013.

Pismeni ispit iz predmeta **Analiza III**

1. Ispitati neprekidnost funkcije $f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

2. (a) Izmjeniti poredak integracije u integralu $\int\limits_0^1 \mathrm{d}y\int\limits_y^{3y} f(x,y)\,\mathrm{d}x$.

(b) Izračunati trojini integral $I=\iiint\limits_G \dfrac{1}{(1+z)^3}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, gdje je oblast G u prvom oktantu ograničena ravnima $x+y=1$, $z=x+y$, $x=0$, $y=0$, $z=0$.

3. Izračunati krivoliniski integral prve vrste

$$I=\oint\limits_C \sqrt{x^2+y^2}\,\mathrm{d}s$$

gdje je C krug $x^2+y^2=ax$, $(a>0)$.

4. Izračunati površinski integral druge vrste

$$I=\iint\limits_S xyz\,\mathrm{d}x\mathrm{d}y$$

gdje je S spoljna strana dijela sfere $x^2+y^2+z^2=1$, $x\geq 0$, $y\geq 0$.

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(#) Ispitati neprekidnost f-je $f(x,y)=\begin{cases} \dfrac{x^2y^3}{2x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

Rj. Jedina tačka u kojoj f-ja $f(x,y)$ može imati prekid je tačka $(0,0)$. F-ja će biti neprekidna u ovoj tački akko

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

tj. akko $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} = 0$.

Posmatrajmo približavanje tački $(0,0)$ preko prave $y=0$:

$$\lim_{(x,0)\to(0,0)} \frac{x^2\cdot 0}{2x^2+0^2} = \lim_{(x,0)\to(0,0)} \frac{0}{2x^2} = 0.$$

Posmatrajmo približavanje tački $(0,0)$ preko niza tački $\left(\frac{1}{n},\frac{1}{n}\right)$, $n\to\infty$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left|\begin{matrix}\text{posmatramo}\\ \text{niz tački}\end{matrix}\right| = \lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot\frac{1}{n^3}}{\frac{2}{n^2}+\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n^5}}{\frac{3}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^5} \begin{matrix}/:n^2\\ /:n^2\end{matrix} = 0.$$

Posmatrajmo približavanje tački $(0,0)$ preko prave $y=mx$:

$$\lim_{(x,mx)\to(0,0)} \frac{x^2\cdot m^3x^3}{2x^2+m^2x^2} \begin{matrix}/:x^2\\ /:x^2\end{matrix} = \lim_{(x,mx)\to(0,0)} \frac{x^3m^3}{2+m^2} = 0.$$

Odavde možemo naslutiti da je vrijednost ovog limesa u tački $(0,0)$ jednaka 0.

ZAŠTO?

$$0 \le \left|\frac{x^2y^3}{2x^2+y^2}\right| = \frac{x^2|y^3|}{2x^2+y^2} \le |y^3| \to 0 \text{ kad } y\to 0$$

tj. kad $(x,y)\to(0,0)$

Prema teoremi "dva policajca" $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} = 0$.

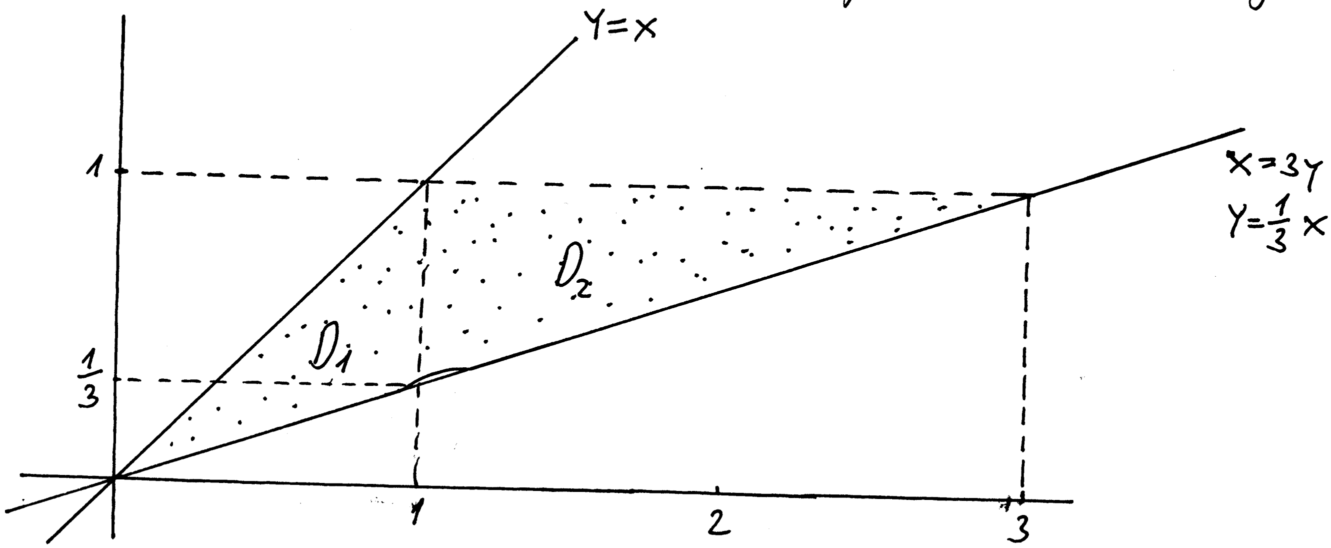
Data f-ja je neprekidna (u svakoj tački).

(#) Izmijeniti poredak integracije u integralu

$$I = \int_0^1 dy \int_y^{3y} f(x,y)\, dx .$$

Rj.

$x = 3y$ i $x = y$ su prave. Skicirajmo oblast integracije



$$D: \begin{cases} 0 \le y \le 1 \\ y \le x \le 3y \end{cases} = D_1 \cup D_2$$

$$D_1: \begin{cases} 0 \le x \le 1 \\ \frac{1}{3}x \le y \le x \end{cases}$$

$$D_2: \begin{cases} 1 \le x \le 3 \\ \frac{1}{3}x \le y \le 1 \end{cases}$$

$$\int_0^1 dy \int_y^{3y} f(x,y)\, dx = \int_0^1 dx \int_{\frac{1}{3}x}^{x} f(x,y)\, dy + \int_1^3 dx \int_{\frac{1}{3}x}^{1} f(x,y)\, dy$$

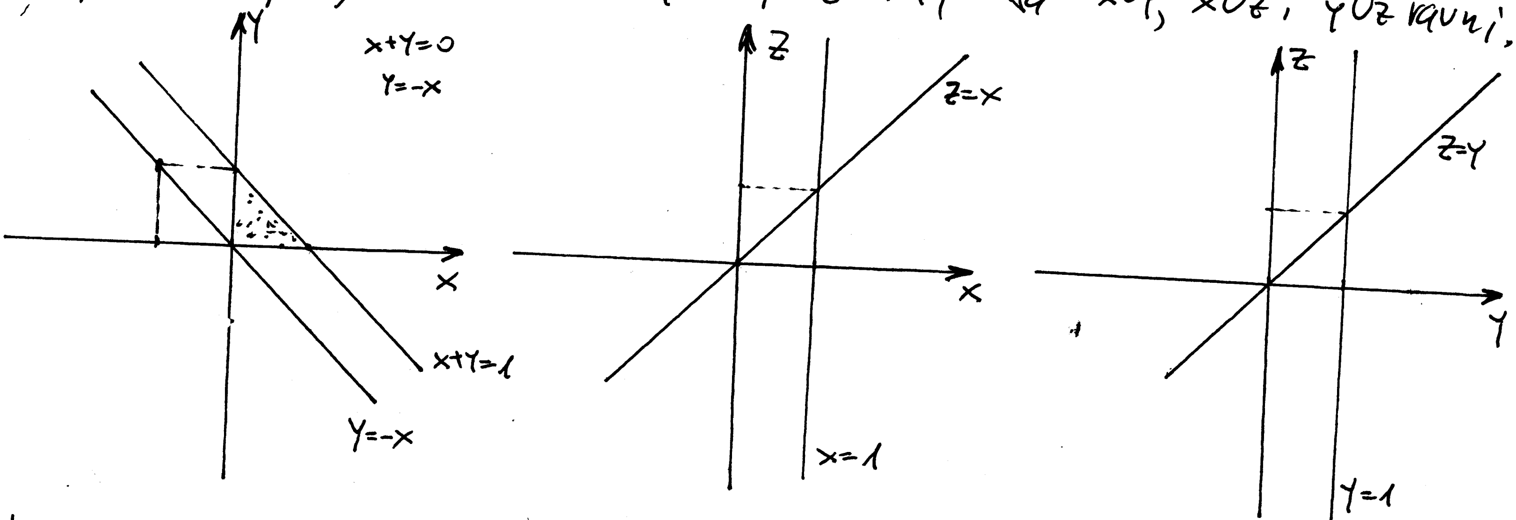
Izračunati trojni integral

$$I=\iiint\limits_G \frac{1}{(1+z)^3}\,dx\,dy\,dz$$

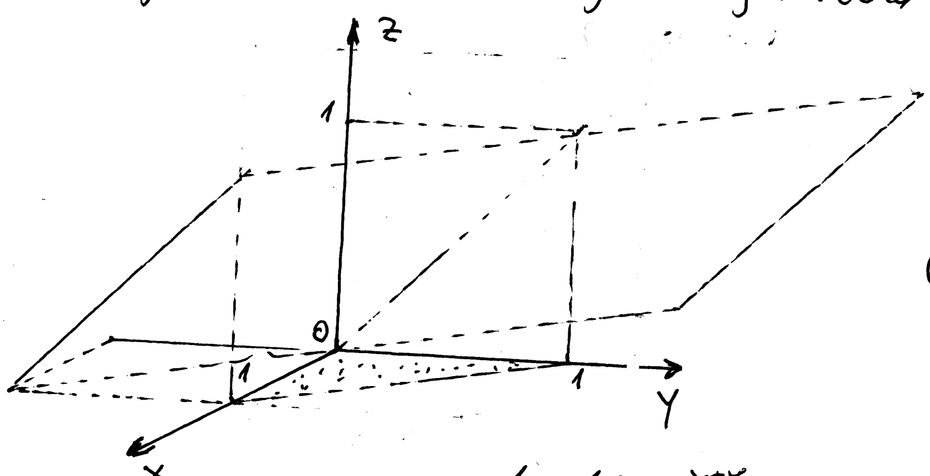
gdje je oblast G u I oktantu ograničena ravnima

$x+y=1,\quad z=x+y,\quad x=0,\quad y=0,\quad z=0.$

Rj. Napravimo presjek ravni $x+y=1$ i $z=x+y$ sa xOy, xOz i yOz ravni.



Iz presjeka vidimo da je ravan $x+y=1$ paralelna sa z osom a da je oblast G odozgo ograničena sa $z=x+y$ ravni



$$G:\begin{cases} 0\le x\le 1\\ 0\le y\le 1-x\\ 0\le z\le x+y\end{cases}$$

$$I=\iiint\limits_G \frac{1}{(1+z)^2}\,dx\,dy\,dz=\int_0^1 dx\int_0^{1-x} dy\int_0^{x+y}(1+z)^{-3}\,d(1+z)=\frac{1}{-2}\int_0^1 dx\int_0^{1-x}\overbrace{(1+z)^{-2}\Big|_0^{x+y}}^{(1+(x+y))^{-2}-1^{-2}}dy=$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}(1+x+y)^{-2}\,d(1+x+y)-\left(-\frac{1}{2}\right)\int_0^1 dx\int_0^{1-x} dy=-\frac{1}{2}\int_0^1(-1)(1+x+y)^{-1}\Big|_0^{1-x}dx+$$

$$+\frac{1}{2}\int_0^1 (1-x)\,dx = \frac{1}{2}\int_0^1 \left(2^{-1}-(1+x)^{-1}\right)dx + \frac{1}{2}\int_0^1 (1-x)\,dx =$$

$$\frac{1}{2} - (1+x)^{-1}$$

$$= \frac{1}{2}\left(\frac{1}{2}x\Big|_0^1 - \ln(1+x)\Big|_0^1 + x\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1\right) =$$

$$= \frac{1}{2}\left(\frac{1}{2} - \ln 2 + 1 - \frac{1}{2}\right) = \frac{1}{2}(1-\ln 2)$$ traženo rješenje

(#) Izračunati krivolinijski integral prve vrste

$$I = \oint_C \sqrt{x^2+y^2}\, ds$$

gdje je C krug $x^2+y^2 = ax$ $(a>0)$.

Rj. Prisjetimo se

<u>Ako je c kriva opisana parametarski</u> $c: \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{cases}$ tada

$$\int_c f(x,y)\, ds = \int_{t_1}^{t_2} f(\eta(t), \mu(t)) \underbrace{\sqrt{(\eta'(t))^2 + (\mu'(t))^2}\, dt}_{ds}$$

$x^2+y^2 = ax$

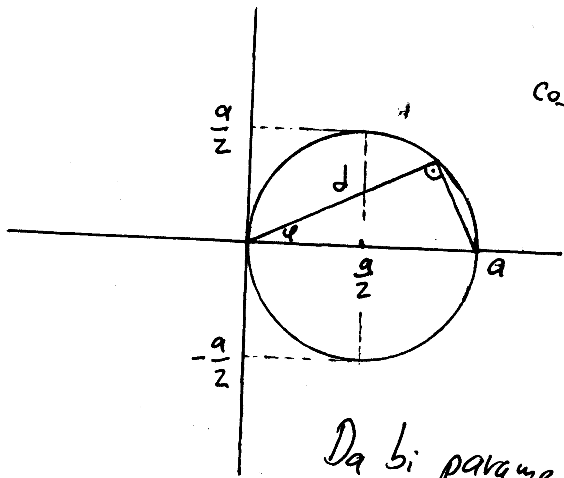
$x^2 - ax + y^2 = 0$

$x^2 - 2 \cdot x \cdot \frac{a}{2} + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$

$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$

krug sa centrom $C\left(\frac{a}{2}, 0\right)$

poluprečnika $r = \frac{a}{2}$



$\cos\varphi = \frac{d}{a}$

$d = a\cos\varphi$

Da bi parametrizirali dati krug pomoći će nam polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

Kako r zavisi od ugla imam

$r = a\cos\varphi$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Parametrizacija datog kruga je

$$\left.\begin{array}{l} x = a\cos\varphi\cos\varphi = a\cos^2\varphi \\ y = a\cos\varphi\sin\varphi = \frac{a}{2}\sin 2\varphi \end{array}\right\} \Rightarrow x^2+y^2 = a^2\cos^2\varphi$$

$$\left.\begin{array}{l} x'_t = 2a\cos\varphi(-\sin\varphi) = -2a\sin\varphi\cos\varphi = -a\sin 2\varphi \\ y'_t = a\cos 2\varphi \end{array}\right\} \Rightarrow \sqrt{{x'_t}^2 + {y'_t}^2} = \sqrt{a^2(\sin^2 2\varphi + \cos^2 2\varphi)} = a$$

$$I = \oint_C \sqrt{x^2+y^2}\, ds = \int_{-\pi/2}^{\pi/2} a\cos\varphi \cdot a\, d\varphi = a^2 \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi = a^2 \sin\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2a^2$$ traženo rješenje

(#) Izračunati površinski integral druge vrste

$$I=\iint_S xyz\,dx\,dy$$

gdje je S spoljna strana dijela sfere $x^2+y^2+z^2=1$, $x\geq 0$, $y\geq 0$.

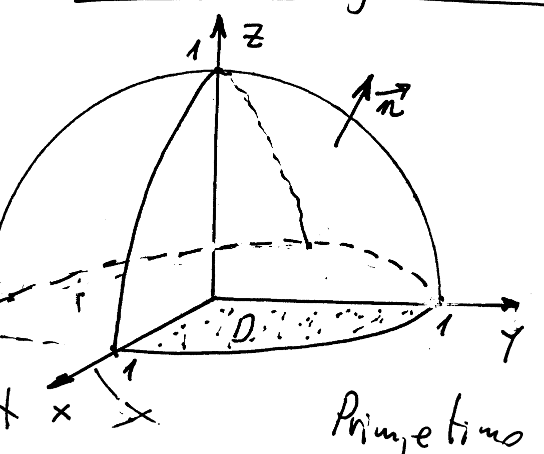
Rj. Prisjetimo se: Neka je površ S data u obliku $z=\eta(x,y)$. Tada

$$\iint_S R(x,y,z)\,dx\,dy = \pm\iint_D R(x,y,\eta(x,y))\,dx\,dy$$ gdje

- $\pm$ zavisi od ugla koji vektor normale zaklapa sa z-osom, npr. $\vec{n}_0=(\cos\alpha, \cos\beta, \cos\gamma)$,

$\cos\gamma>0$	$\Rightarrow$	$+$
$\cos\gamma<0$	$\Rightarrow$	$-$
$\cos\gamma=0$	$\Rightarrow$	0

- D je ortogonalna projekcija površi S na xOy ravan



$$z^2=1-x^2-y^2$$

$$z=\pm\sqrt{1-x^2-y^2}$$

kako je $x\geq 0$, $y\geq 0$ to je $z=\sqrt{1-x^2-y^2}$

Primjetimo da je $0<\gamma<90^\circ \Rightarrow \cos\gamma>0$

$$\iint_S xyz\,dx\,dy=\iint_D xy\sqrt{1-x^2-y^2}\,dx\,dy=\left|\begin{array}{ll}\text{uvedimo polarne koordinate} & \\ x=r\cos\varphi & 1-x^2-y^2=1-r^2 \\ y=r\sin\varphi & \\ dx\,dy=r\,dr\,d\varphi & D\xrightarrow{\text{transf.}}D':\begin{cases}0\leq r\leq 1\\ 0\leq\varphi\leq\frac{\pi}{2}\end{cases}\end{array}\right|$$

$$=\iint_{D'} r^3\sin\varphi\cos\varphi\sqrt{1-r^2}\,dr\,d\varphi=\int_0^1 r^3\sqrt{1-r^2}\,dr\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi\overset{\text{za vježbu}}{=}\ldots=\frac{2}{15}\cdot\frac{1}{2}=\frac{1}{15}$$